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Two Optimal Allocations under Management Systems: Game-theoretical Approaches

Hui-Chuan Wei[1], *Pang-Tung Liu*[2], *Jen-Ning Liou*[3] *and Yu-Hsien Liao*[3]

[1]National Chung Cheng University, Taiwan,
[2]National Yunlin University of Science and Technology
and [3]National Pingtung University

Abstract

By applying the notion of the efficient Banzhaf index, any additional fixed utility should be distributed equally among the players who are concerned. In many applications, however, this notion seems unrealistic for the situation that is being modeled. Therefore, inspired by the notion of the weighted allocation of non-separable costs (WANSC), we adopt weight functions to introduce a modification of the efficient Banzhaf index, which we name the weighted Banzhaf index. In order to present the rationality of the weighted Banzhaf index, we adopt some reasonable properties to characterize the weighted Banzhaf index. Based on different viewpoints, we further define excess functions to propose alternative formulations and related dynamic processes for the weighted Banzhaf index and the WANSC respectively.

Keywords: Weight function, the weighted Banzhaf index, excess function, dynamic process.

1. Introduction

A member in a voting system is, e.g., a party in a parliament or a country in a confederation. In general, each member will have a certain number of votes, and so its power will be different. *The marginal index and the Banzhaf index* [2] are two quantities to measure the political power of each member of a voting system. Related results can be found in, e.g., Algaba et al. [1], Dubey and Shapley [3], Haller [4], Hwang and Liao [7], Lehrer [8], Liao [10], Moulin [13], Owen [14], Ruiz [16], and so on. It is known that each of these two indexes does not necessarily distribute the entire utility over all players in the grand coalition. Thus, two efficient extensions of these two indexes, the *equal allocation of non-separable costs* (EANSC, Ransmeier [15]) and the *efficient Banzhaf index* (Hwang and Liao [6]), are proposed respectively.

Based on the notion of the EANSC, all players firstly receive their marginal contributions from the grand coalition, and further allocate the remaining utilities equally. Based on the notion of the efficient Banzhaf index, all players firstly receive their marginal contributions from all coalitions, and further allocate the remaining utilities equally. Based on these two indexes, any additional fixed utility (e.g., the cost of a common facility) should be distributed equally among the players who are concerned. In many applications, however, these two indexes seems unrealistic for the situation that is being modeled. Players might represent constituencies of different sizes; players might have different bargaining abilities. Also, lack of symmetry may arise when different bargaining abilities for different players are modeled. In line with the above interpretations, we would now desire that any additional fixed utility could be distributed among the players in proportion to their weights. In various applications of transferable-utility (TU) games it seems to be natural to assume that the players are given some *a-priori measures of importance, called weights*. For example, when we deal with a problem of utility allocation among investment projects, then the weights could be associated to the profitability of the different projects. In a question of allocating travel costs among various places visited, the weights could be the number of days spent at each one (cf., Shapley [19]).

In order to modify the discrimination among players, Shapley [17] proposed the *weighted Shapley value*. The weighted Shapley values attempt to define a fair way of dividing up the worth of the grand coalition by assigning to each player a weighted average of the marginal contributions he makes to all possible coalitions. Subsequently, Kalai and Samet [11] introduced the notion of a *weight system*, allowing a weight of zero for some of the players. They also gave a new weighted extension of the Shapley value, using a random order approach. Later, Liao et al. [9] adopted the *weight functions* to propose the *weighted allocation of non-separable costs* (WANSC). Based on the WANSC, all players firstly receive their marginal contributions from the grand coalitions, and further allocate the remaining utilities proportionally by applying weights. Here we propose different results as follows.

(1) Similar to the notion of the WANSC, we adopt weight functions to propose the *weighted Banzhaf index* in Section 2. Further, we characterize the weighted Banzhaf index by means of the *efficiency-average-reduced game*.
(2) In Sections 2 and 3, we present alternative formulations for the weighted Banzhaf index and the WANSC in terms of *excess functions*. The excess of a coalition could be treated as the *variation* between the productivity and the total payoff of the coalition.
(3) Based on excess functions, we also propose dynamic processes to illustrate that the weighted Banzhaf index and the WANSC can be approached by players who start from an arbitrary efficient payoff vector. When a player withdraws from the coalitions he/she/it joined, several types of complaints may be occurred from other players. These dynamic processes are devoted to regulating these complaints to be more coincident among all players.

In Sections 4 and 5, some more discussions and interpretations are also provided in detail.

2. The Weighted Banzhaf index and Related Results

A coalitional game with transferable-utility (TU game) is a pair (N, v) where N is the grand coalition and v is a mapping such that $v : 2^N \to \mathbb{R}$ and $v(\emptyset) = 0$. Denote the class of all TU games by G. A **solution** on G is a function ψ which associates with each game $(N, v) \in G$ an element $\psi(N, v)$ of $\mathbb{R}^N$. A solution ψ satisfies **efficiency** (**EFF**) if $\sum_{i\in N} \psi_i(N, v) = v(N)$ for all $(N, v) \in G$. The Banzhaf index and the efficient Banzhaf index are defined as follows.

Definition 1. The **efficient Banzhaf index** (Hwang and Liao [6]), $\overline{\mu}$, is the solution on G which associates with $(N, v) \in G$ and each player $i \in N$ the value

$$\overline{\mu_i}(N, v) = \mu_i(N, v) + \frac{1}{|N|} \cdot [v(N) - \sum_{k\in N} \mu_k(N, v)], \tag{2.1}$$

where $\mu_i(N, v) = \frac{1}{2^{|N|-1}} \sum_{\substack{S\subseteq N \\ i\in S}} [v(S) - v(S\setminus\{i\})]$ is the **Banzhaf index** (Banzhaf [2]) of i. It is known that the Banzhaf index violates EFF, and the efficient Banzhaf index satisfies EFF.

Let $(N, v) \in G$. A function $w : N \to \mathbb{R}^+$ is called a **weight function** if w is a non-negative function. In different situations, players in N could be assigned different weights by weight functions. These weights could be interpreted as *a-priori measures of importance*; they are taken to reflect considerations not captured by the characteristic function. For example, we may be dealing with a problem of cost allocation among investment projects. Then the weights could be associated to the profitability of the different projects. In a problem of allocating travel costs among various institutions visited (cf., Shapley [19]), the weights may be the number of days spent at each one.

Given $(N, v) \in G$ and a weight function w, we define $|S|_w = \sum_{i\in S} w(i)$ for all $S \subseteq N$. The weighted Banzhaf index is defined as follows.

Definition 2. Let w be a weight function. The **weighted Banzhaf index**, $\overline{\mu^w}$, is the solution on G which associates with $(N, v) \in G$ and all players $i \in N$ the value

$$\overline{\mu_i^w}(N, v) = \mu_i(N, v) + \frac{w(i)}{|N|_w} \cdot [v(N) - \sum_{k\in N} \mu_k(N, v)]. \tag{2.2}$$

By the definition of $\overline{\mu^w}$, all players firstly receive their marginal contributions from all coalitions, and further allocate the remaining utilities proportionally by applying weights.

Lemma 1. *The weighted Banzhaf index $\overline{\mu^w}$ satisfies EFF.*

Proof. Let $(N, v) \in G$. By Definition 2,

$$\begin{aligned}
\sum_{i\in N} \overline{\mu_i^w}(N, v) &= \sum_{i\in N} \mu_i(N, v) + \frac{w(i)}{|N|_w} \cdot [v(N) - \sum_{k\in N} \mu_k(N, v)] \\
&= \sum_{i\in N} \mu_i(N, v) + \sum_{i\in N} \frac{w(i)}{|N|_w} \cdot [v(N) - \sum_{k\in N} \mu_k(N, v)]
\end{aligned}$$

$$= \sum_{i\in N} \mu_i(N,v) + \frac{|N|_w}{|N|_w} \cdot [v(N) - \sum_{k\in N} \mu_k(N,v)] = v(N).$$

Hence, the weighted Banzhaf index $\overline{\mu^w}$ satisfies EFF.

Next, we present an alternative formulation for the weighted Banzhaf index in terms of *excess functions*. If $x \in \mathbb{R}^N$ and $S \subseteq N$, write x_S for the restriction of x to S and write $x(S) = \sum_{i\in S} x_i$. Denote that $X(N,v) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N)\}$ for all $(N,v) \in G$. The **excess** of a coalition $S \subseteq N$ at x is the real number $e(S,v,x) = v(S) - x(S)$.

Lemma 2. *Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. Then*

$$\begin{aligned} & w(j) \sum_{S\subseteq N\{i\}} [e(S,v,x) - e(S\cup\{i\},v,x)] \\ & = w(i) \sum_{S\subseteq N\{j\}} [e(S,v,x) - e(S\cup\{j\},v,x)] \qquad \forall\ i,j \in N \\ \Leftrightarrow \quad & x = \overline{\mu^w}(N,v).) \end{aligned}$$

Proof. Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. For all $i,j \in N$,

$$\begin{aligned} & w(j) \sum_{S\subseteq N\{i\}} [e(S,v,x) - e(S\cup\{i\},v,x)] \\ = & w(i) \sum_{S\subseteq N\{j\}} [e(S,v,x) - e(S\cup\{j\},v,x)] \\ & \Leftrightarrow \\ & w(j) \sum_{S\subseteq N\{i\}} [v(S) - x(S) + x(S\cup\{i\}) - v(S\cup\{i\})] \\ = & w(i) \sum_{S\subseteq N\{j\}} [v(S) - x(S) + x(S\cup\{j\}) - v(S\cup\{j\})] \\ & \Leftrightarrow \\ & w(j) \sum_{S\subseteq N\{i\}} [x_i - v(S\cup\{i\}) + v(S)] \qquad (2.3) \\ = & w(i) \sum_{S\subseteq N\{j\}} [x_j - v(S\cup\{j\}) + v(S)] \\ & \Leftrightarrow \\ & 2^{|N|-1} w(j)[x_i - \frac{1}{2^{|N|-1}} \sum_{S\subseteq N\{i\}} [v(S\cup\{i\}) - v(S)]] \\ = & 2^{|N|-1} w(i)[x_j - \frac{1}{2^{|N|-1}} \sum_{S\subseteq N\{j\}} [v(S\cup\{j\}) - v(S)]] \\ & \Leftrightarrow \\ & w(j) \cdot [x_i - \mu_i(N,v)] = w(i) \cdot [x_j - \mu_j(N,v)]. \end{aligned}$$

By Definition 2,

$$w(j) \cdot [\overline{\mu_i^w}(N,v) - \mu_i(N,v)] = w(i) \cdot [\overline{\mu_j^w}(N,v) - \mu_j(N,v)]. \tag{2.4}$$

By equations (2.3) and (2.4),

$$[x_i - \overline{\mu_i^w}(N,v)] \sum_{j\in N} w(j) = w(i) \sum_{j\in N} [x_j - \overline{\mu_j^w}(N,v)].$$

Since $x \in X(N,v)$ and $\overline{\mu^w}$ satisfies EFF,

$$[x_i - \overline{\mu_i^w}(N,v)] \cdot |N|_w = w(i) \cdot [v(N) - v(N)] = 0.$$

Therefore, $x_i = \overline{\mu_i^w}(N,v)$ for all $i \in N$.

Subsequently, we adopt the efficiency-average-reduced game to characterize the weighted Banzhaf index.

Definition 3 (Hwang and Liao [6])**.** Let $(N,v) \in G$, $S \subseteq N$ and ψ be a solution. The **efficiency-average-reduced game $(S, v_{S,\psi})$ with respect to ψ and S** is defined by

$$v_{S,\psi}(T) = \begin{cases} 0 & T = \emptyset, \\ v(N) - \sum\limits_{i\in N\setminus S} \psi_i(N,v) & T = S, \\ \dfrac{1}{2^{|N\setminus S|}} \sum\limits_{Q\subseteq N\setminus S} [v(T\cup Q) - \sum_{i\in Q} \psi_i(N,v)] & \text{Otherwise.} \end{cases}$$

The efficiency-average-reduction says that given a proposed payoff vector $\psi(N,v)$, the worth of a coalition T in $(S, v_{S,\psi})$ is computed under the assumption that T can secure the cooperation of any subgroup Q of $N\setminus S$, provided each member of Q receives his component of $\psi(N,v)$. After these payments are made, what remains for T is the value $v(T\cup Q) - \sum_{i\in Q} \psi_i(N,v)$. Averaging behavior on the part of T involves finding the average of the values $v(T\cup Q) - \sum_{i\in Q} \psi_i(N,v)$ for all $Q \subseteq N\setminus S$. A solution ψ satisfies **bilateral efficiency-average-consistency** (**BEACON**) if $\psi_i(S, v_{S,\psi}) = \psi_i(N,v)$ for all $(N,v) \in G$ with $|N| \geq 2$, for all $S \subseteq N$ with $|S| = 2$ and for all $i \in S$.

Lemma 3. *The weighted Banzhaf index $\overline{\mu^w}$ satisfies BEACON.*

Proof. Let $(N,v) \in G$, $S \subseteq N$ with $|S| = 2$ and w be a weight function. Let $x = \overline{\mu^w}(N,v)$. Suppose $S = \{i,j\}$ then

$$\begin{aligned} &\sum_{T\subseteq S\setminus\{i\}} [e(T, v_{S,\overline{\mu^w}}, x_S) - e(T\cup\{i\}, v_{S,\overline{\mu^w}}, x_S)] \\ &= [e(\{j\}, v_{S,\overline{\mu^w}}, x_S) - e(S, v_{S,\overline{\mu^w}}, x_S)] + [e(\emptyset, v_{S,\overline{\mu^w}}, x_S) - e(\{i\}, v_{S,\overline{\mu^w}}, x_S)] \\ &= (v_{S,\overline{\mu^w}}(\{j\}) - x_j) - (v_{S,\overline{\mu^w}}(S) - x_S(S)) + 0 - (v_{S,\overline{\mu^w}}(\{i\}) - x_i) \\ &= (v_{S,\overline{\mu^w}}(\{j\}) - x_j) - 0 + 0 - (v_{S,\overline{\mu^w}}(\{i\}) - x_i) \end{aligned}$$

$$
\begin{aligned}
&= \Big(\Big[\frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} [v(\{j\}\cup Q) - \sum_{k\in Q} x_k]\Big] - x_j\Big) \\
&\quad -\Big(\Big[\frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} [v(\{i\}\cup Q) - \sum_{k\in Q} x_k]\Big] - x_i\Big) \\
&= \frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} ([v(\{j\}\cup Q) - \sum_{k\in Q} x_k] - [v(\{i\}\cup Q) - \sum_{k\in Q} x_k - x_i]) \\
&= \frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} ([v(\{j\}\cup Q) - \sum_{k\in \{j\}\cup Q} x_k] - [v(\{i\}\cup Q) - \sum_{k\in \{j\}\cup Q} x_k]) \\
&= \frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus S} [(e(\{j\}\cup Q, v, x) - (e(\{i\}\cup Q, v, x)] \\
&= \frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus \{i,j\}} [(e(\{j\}\cup Q, v, x) - (e(\{i\}\cup Q, v, x)] \\
&= \frac{1}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus \{i\}} [(e(Q, v, x) - (e(Q\cup\{i\}, v, x)] \qquad (2.5)
\end{aligned}
$$

By EFF of $\overline{\mu^w}$ and the definition of efficiency-average-reduced game, $x_S \in X(S, v_{S,\overline{\mu^w}})$. In addition, by Equation (2.5) and Lemma 2,

$$
\begin{aligned}
& w(j) \cdot \sum_{T\subseteq S\setminus\{i\}} [e(T, v_{S,\overline{\mu^w}}, x_S) - e(T\cup\{i\}, v_{S,\overline{\mu^w}}, x_S)] \\
&= \frac{w(j)}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus\{i\}} [(e(Q, v, x) - (e(Q\cup\{i\}, v, x)] \qquad \text{(by Equation (2.5))} \\
&= \frac{w(j)}{2^{|N\setminus S|}} \cdot \sum_{Q\subseteq N\setminus\{j\}} [(e(Q, v, x) - (e(Q\cup\{j\}, v, x)] \qquad \text{(by Lemma 2)} \\
&= w(j) \cdot \sum_{T\subseteq S\setminus\{j\}} [e(T, v_{S,\overline{\mu^w}}, x_S) - e(T\cup\{j\}, v_{S,\overline{\mu^w}}, x_S)]. \qquad \text{(similar to Equation (2.5))}
\end{aligned}
$$

By Lemma 2 and $x_S \in X(S, v_{S,\overline{\mu^w}})$, we have that $x_S = \overline{\mu^w}(S, v_{S,\overline{\mu^w}})$. Hence, $\overline{\mu^w}$ satisfies BEACON.

Inspired by Hart and Mas-Colell [5], we provide an axiomatic result of the weighted Banzhaf index as follows. A solution ψ satisfies **weighted-Banzhaf standard for games** (**WBSFG**) if $\psi(N, v) = \overline{\mu^w}(N, v)$ for all $(N, v) \in G$ with $|N| \leq 2$.

Lemma 4. *If a solution ψ satisfies WBSFG and BEACON, then it satisfies EFF.*

Proof. Suppose ψ satisfies WBSFG and BEACON. Let $(N, v) \in G$. If $|N| \leq 2$, then ψ satisfies EFF by BEACON of ψ. Suppose $|N| > 2$, $i, j \in N$ and $S = \{i, j\}$. Since ψ satisfies EFF in two-person games,

$$\psi_i(S, v_{S,\psi}) + \psi_j(S, v_{S,\psi}) = v_{S,\psi}(S) = v(N) - \sum_{k\neq i,j} \psi_k(N, v). \qquad (2.6)$$

By BEACON of ψ,

$$\psi_t(S, v_{S,\psi}) = \psi_t(N, v) \quad \text{for all } t \in S. \tag{2.7}$$

By equations (2.6) and (2.7), $v(N) = \sum_{k\in N} \psi_k(N, v)$, i.e., ψ satisfies EFF.

Theorem 1. *A solution ψ satisfies WBSFG and BEACON if and only if $\psi = \overline{\mu^w}$.*

Proof. By Lemma 3, $\overline{\mu^w}$ satisfies BEACON. Clearly, $\overline{\mu^w}$ satisfies WBSFG.

To prove uniqueness, suppose ψ satisfies WBSFG and BEACON. By Lemma 4, ψ satisfies EFF. Let $(N, v) \in G$. If $|N| \le 2$, it is trivial that $\psi(N, v) = \overline{\mu^w}(N, v)$ by SFG. Assume that $|N| > 2$. Let $i \in N$ and $S = \{i, j\}$ for some $j \in N\backslash\{i\}$. Then

$$\begin{aligned}\psi_i(N, v) - \overline{\mu_i^w}(N, v) &= \psi_i(S, v_{S,\psi}) - \overline{\mu_i^w}(S, v_{S,\overline{\mu^w}}) \quad \text{(by BEACON of } \psi, \overline{\mu^w})\\ &= \overline{\mu_i^w}(S, v_{S,\psi}) - \overline{\mu_i^w}(S, v_{S,\overline{\mu^w}}) \quad \text{(by WBSFG of } \psi, \overline{\mu^w})\\ &= \frac{1}{2}[v_{S,\psi}(S) + v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\})]\\ &\quad -\frac{1}{2}[v_{S,\overline{\mu^w}}(S) + v_{S,\overline{\mu^w}}(\{i\}) - v_{S,\overline{\mu^w}}(\{j\})].\end{aligned} \tag{2.8}$$

By definitions of $v_{S,\psi}$ and $v_{S,\overline{\mu^w}}$,

$$\begin{aligned}v_{S,\psi}(\{i\}) - v_{S,\psi}(\{j\}) &= \frac{1}{2^{|N\backslash S|}} \cdot \sum_{Q\subseteq N\backslash S} [v(\{i\} \cup Q) - v(\{j\} \cup Q)]\\ &= v_{S,\overline{\mu^w}}(\{i\}) - v_{S,\overline{\mu^w}}(\{j\}).\end{aligned} \tag{2.9}$$

By equations (2.8) and (2.9),

$$\begin{aligned}\psi_i(N, v) - \overline{\mu_i^w}(N, v) &= \frac{1}{|S|_w} \cdot [v_{S,\psi}(S) - v_{S,\overline{\mu^w}}(S)]\\ &= \frac{1}{|S|_w} \cdot [\psi_i(N, v) + \psi_j(N, v) - \overline{\mu_i^w}(N, v) - \overline{\mu_j^w}(N, v)].\end{aligned}$$

That is,

$$[\psi_i(N, v) - \overline{\mu_i^w}(N, v)] = [\psi_j(N, v) - \overline{\mu_j^w}(N, v)].$$

By EFF of ψ and $\overline{\mu^w}$,

$$0 = v(N) - v(N) = \sum_{j\in N} [\psi_j(N, v) - \overline{\mu_j^w}(N, v)] = |N| \cdot [\psi_i(N, v) - \overline{\mu_i^w}(N, v)].$$

Hence, $\psi_i(N, v) = \overline{\mu_i^w}(N, v)$ for all $i \in N$.

The following examples are to show that each of the axioms used in Theorem 1 is logically independent of the remaining axioms.

Example 1. Define a solution ψ by for all $(N, v) \in G$ and for all $i \in N$,

$$\psi_i(N, v) = \frac{v(N)}{|N|}.$$

Clearly, ψ satisfies BEACON, but it violates WBSFG.

Example 2. Define a solution ψ by for all $(N, v) \in G$ and for all $i \in N$,

$$\psi_i(N, v) = \begin{cases} \overline{\mu_i^w}(N, v), & \text{if } |N| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, ψ satisfies WBSFG, but it violates BEACON.

By applying a specific reduction, Maschler and Owen [12] defined a correction function to introduce a dynamic process for the Shapley value [18]. Different from the notion due to Maschler and Owen [12], we adopt excess function to propose an alternative correction function and related dynamic process for the weighted Banzhaf index.

Definition 4. Let $(N, v) \in G$, $i \in N$ and w be a weight function. The correction function $f_i^{\overline{\mu^w}} : X(N, v) \to \mathbb{R}$ is defined by

$$\begin{aligned} f_i^{\overline{\mu^w}}(x) = x_i + t \sum_{j \in N \setminus \{i\}} \Big(& w(i) \sum_{S \subseteq N\{j\}} [e(S, v, x) - e(S \cup \{j\}, v, x)] \\ & -w(j) \sum_{S \subseteq N\{i\}} [e(S, v, x) - e(S \cup \{i\}, v, x)] \Big), \end{aligned}$$

where $t \in (0, \infty)$, which reflects the assumption that player i does not ask for full correction (when $t = 1$) but only (usually) a fraction of it.

The correction function is based on the idea that, each agent shortens the weighted excess relating to his own and others' non-participation in all coalitions, and adopts these regulations to correct the original payoff.

The following lemma shows that the correction function is well-defined, i.e., the efficiency is preserved under the correction function.

Lemma 5. *Let $(N, v) \in G$, w be a weight function and $f^{\overline{\mu^w}} = (f_i^{\overline{\mu^w}})_{i \in N}$. If $x \in X(N, v)$, then $f^{\overline{\mu^w}}(x) \in X(N, v)$.*

Proof. Let $(N, v) \in G$, $i, j \in N$, $x \in X(N, v)$ and w be a weight function. Similar to the equation (2.3),

$$\begin{aligned} & w(i) \sum_{S \subseteq N\{j\}} [e(S, v, x) - e(S \cup \{j\}, v, x)] - w(j) \sum_{S \subseteq N\{i\}} [e(S, v, x) - e(S \cup \{i\}, v, x)] \\ & = w(i)[x_j - \overline{\mu_j^w}(N, v)] - w(j)[x_i - \overline{\mu_i^w}(N, v)]. \end{aligned} \tag{2.10}$$

By equation (2.10),

$$\begin{aligned} \sum_{j \in N \setminus \{i\}} (& w(i) \sum_{S \subseteq N\{j\}} [e(S, v, x) - e(S \cup \{j\}, v, x)] \\ & -w(j) \sum_{S \subseteq N\{i\}} [e(S, v, x) - e(S \cup \{i\}, v, x)]) \end{aligned}$$

$$
\begin{aligned}
&= w(i) \sum_{j\in N\setminus\{i\}} [x_j - \overline{\mu^w_j}(N,v)] - [x_i - \overline{\mu^w_i}(N,v)] \sum_{j\in N\setminus\{i\}} w(j) \\
&= w(i)[v(N) - v(N)] - [x_i - \overline{\mu^w_i}(N,v)]|N|_w \quad \text{(by EFF of } \overline{\mu^w}, x \in X(N,v)) \\
&= |N|_w(\overline{\mu^w_i}(N,v) - x_i). \qquad (2.11)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\sum_{i\in N} |N|_w(\overline{\mu^w_i}(N,v) - x_i) &= |N|_w(v(N) - v(N)) \quad \text{(by EFF of } \overline{\mu^w}, x \in X(N,v)) \\
&= 0.. \qquad (2.12)
\end{aligned}
$$

So we have that

$$
\begin{aligned}
\sum_{i\in N} f_i^{\overline{\mu^w}}(x) &= \sum_{i\in N} [x_i + t \sum_{j\in N\setminus\{i\}} (w(i) \sum_{S\subseteq N\{j\}} [e(S,v,x) - e(S\cup\{j\},v,x)] \\
&\quad -w(j) \sum_{S\subseteq N\{i\}} [e(S,v,x) - e(S\cup\{i\},v,x)])] \\
&= v(N) + t\cdot 0 \quad \text{(by equations (2.11), (2.12) and } x \in X(N,v)) \\
&= v(N).
\end{aligned}
$$

Hence, $f^{\overline{\mu^w}}(x) \in X(N,v)$ if $x \in X(N,v)$.

Based on Lemma 5, we can define $x^0 = x, x^1 = f^{\overline{\mu^w}}(x^0), \ldots, x^q = f^{\overline{\mu^w}}(x^{q-1})$ for all $(N,v) \in G$, for all $x \in X(N,v)$ and for all $q \in N$. Next, we adopt the correction function to propose a dynamic process.

Theorem 2. *Let $(N,v) \in G$ and w be a weight function. If $0 < t < \dfrac{2}{|N|_w}$, then $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\mu^w}(N,v)$ for all $x \in X(N,v)$.*

Proof. Let $(N,v) \in G$, $i \in N$, $x \in X(N,v)$ and w be a weight function. By equation (2.11) and definition of $f^{\overline{\mu^w}}$,

$$
\begin{aligned}
f_i^{\overline{\mu^w}}(x) - x_i &= t \sum_{j\in N\setminus\{i\}} (w(i) \sum_{S\subseteq N\{j\}} [e(S,v,x) - e(S\cup\{j\},v,x)] \\
&\quad -w(j) \sum_{S\subseteq N\{i\}} [e(S,v,x) - e(S\cup\{i\},v,x)]) \\
&= t\cdot |N|_w(\overline{\mu^w_i}(N,v) - x_i).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\overline{\mu^w_i}(N,v) - f_i^{\overline{\mu^w}}(x) &= \overline{\mu^w_i}(N,v) - x_i + x_i - f_i^{\overline{\mu^w}}(x) \\
&= \overline{\mu^w_i}(N,v) - x_i - t\cdot |N|_w \cdot (\overline{\mu^w_i}(N,v) - x_i) \\
&= (1 - t\cdot |N|_w)[\overline{\mu^w_i}(N,v) - x_i].
\end{aligned}
$$

For all $q \in \mathbb{N}$,

$$\overline{\mu^w}(N,v) - x^q = (1 - t \cdot |N|_w)^q [\overline{\mu^w}(N,v) - x].$$

If $0 < t < \frac{2}{|N|_w}$, then $-1 < (1 - t \cdot |N|_w) < 1$ and $\{x^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\mu^w}(N,v)$.

3. The WANSC and Related Results

In this section, we introduce an excess formulation and related dynamic process for the WANSC.

Definition 5.

- The **equal allocation of non-separable costs** (**EANSC**, Ransmeier [15]), $\overline{\beta}$, is the solution which associates with $(N,v) \in G$ and each player $i \in N$ the value

$$\overline{\beta_i}(N,v) = \beta_i(N,v) + \frac{1}{|N|} \cdot [v(N) - \sum_{k \in N} \beta_k(N,v)], \tag{3.1}$$

  where $\beta_i(N,v) = v(N) - v(N \backslash \{i\})$ is the **marginal index** of i.
- The **weighted allocation of non-separable costs** (**WANSC**, Liao et al. [9]), $\overline{\beta^w}$, is the solution which associates with $(N,v) \in G$ and all players $i \in N$ the value

$$\overline{\beta_i^w}(N,v) = \beta_i(N,v) + \frac{w(i)}{|N|_w} \cdot [v(N) - \sum_{k \in N} \beta_k(N,v)]. \tag{3.2}$$

Different from the results proposed by Liao et al. [9], we present an alternative formulation for the WANSC in terms of excess.

Lemma 6. *Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. Then*

$$w(j) \cdot e(N \backslash \{i\}, v, x) = w(i) \cdot e(N \backslash \{j\}, v, x) \quad \forall\ i,j \in N \Leftrightarrow x = \overline{\beta^w}(N,v).$$

Proof. Let $(N,v) \in G$, $x \in X(N,v)$ and w be a weight function. For all $i,j \in N$,

$$\begin{aligned} & w(j) \cdot e(N \backslash \{i\}, v, x) = w(i) \cdot e(N \backslash \{j\}, v, x) \\ \Leftrightarrow\ & w(j)[v(N \backslash \{i\}) - x(N \backslash \{i\})] = w(i)[v(N \backslash \{j\}) - x(N \backslash \{j\})] \\ \Leftrightarrow\ & w(j)[v(N \backslash \{i\}) - v(N) + x_i] = w(i)[v(N \backslash \{j\}) - v(N) + x_j] \\ \Leftrightarrow\ & w(j) \cdot [x_i - \beta_i(N,v)] = w(i) \cdot [x_j - \beta_j(N,v)] \end{aligned} \tag{3.3}$$

Based on (3.2),

$$w(j) \cdot [\overline{\beta_i^w}(N,v) - \beta_i(N,v)] = w(i) \cdot [\overline{\beta_j^w}(N,v) - \beta_j(N,v)]. \tag{3.4}$$

By equations (3.3) and (3.4),

$$w(j) \cdot [x_i - \overline{\beta_i^w}(N,v)] = w(i) \cdot [x_j - \overline{\beta_j^w}(N,v)].$$

Thus, $[x_i - \overline{\beta^w_i}(N,v)]\sum_{j\in N} w(j) = w(i)\sum_{j\in N}[x_j - \overline{\beta^w_j}(N,v)]$. Since $x \in X(N,v)$ and $\overline{\beta^w}$ satisfies EFF,

$$[x_i - \overline{\beta^w_i}(N,v)] \cdot |N|_w = w(i) \cdot [v(N) - v(N)] = 0.$$

Therefore, $x_i = \overline{\beta^w_i}(N,v)$ for all $i \in N$.

Similar to Section 2, we adopt excess functions to define another correction function, and show that this correction function is well-defined.

Definition 6. Let $(N,v) \in G$, $i \in N$ and w be a weight function. The correction function $f_i^{\overline{\beta^w}} : X(N,v) \to \mathbb{R}$ is defined by

$$f_i^{\overline{\beta^w}}(x) = x_i + t \sum_{j\in N\setminus\{i\}} (w(i)\cdot e(N\setminus\{j\},v,x) - w(j)\cdot e(N\setminus\{i\},v,x)),$$

where $t \in (0,\infty)$.

Lemma 7. *Let $(N,v) \in G$, w be a weight function and $f^{\overline{\beta^w}} = (f_i^{\overline{\beta^w}})_{i\in N}$. If $x \in X(N,v)$, then $f^{\overline{\beta^w}}(x)X(N,v)$.*

Proof. The proof is similar to Lemma 5. Hence, we omit it.

Based on Lemma 7, we can define $y^0 = y$, $y^1 = f^{\overline{\beta^w}}(y^0), \ldots, y^q = f^{\overline{\beta^w}}(y^{q-1})$ for all $(N,v) \in G$, for all $y \in X(N,v)$ and for all $q \in \mathbb{N}$. Subsequently, we adopt the correction function to propose a dynamic process for the WANSC.

Theorem 3. *Let $(N,v) \in G$ and w be a weight function. If $0 < t < \dfrac{2}{|N|_w}$, then $\{y^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\beta^w}(N,v)$ for all $y \in X(N,v)$.*

Proof. The proof is similar to Theorem 2. Hence, we omit it.

4. Illustration and Application

In this section, we provide illustration and application of TU games, the weighted Banzhaf index and the WANSC in the setting of "utilities allocation for warehouse management systems", such as the COSTCO, the Carrefour and so on.

In a warehouse organization, such as the COSTCO, each department of the warehouse organization may take its operation strategies to manage. Besides competing in merchandising, all departments should develop to raise entire utilities of whole the warehouse organization also, such as security department, logistics department, purchasing department and so on. This kind of utilities allocation problem could be formulated as follows: Let $N = \{1, 2, \ldots, n\}$ be a collection of all departments of the warehouse organization that could be provided jointly by some coalitions $S \subseteq N$ and let $v(S)$ be the profit of providing the strategical coalition $S \subseteq N$ jointly. Each coalition $S \subseteq N$ could be formed by focusing on specific operational goal. The function v could be treated as an utility function which assigns to each operation strategical coalition $S \subseteq N$ the worth that the coalition S can obtain. Modeled in this way, the utilities allocation management system of a warehouse organization could be considered as a cooperative TU game, with v being its characteristic function. As mentioned in above sections, however, it may not be appropriate in many situations if any additional fixed utility should be distributed equally among the players who are concerned. Thus, it is reasonable that weights are assigned to players and any fixed utility should be divided according to these weights. Therefore, the weighted Banzhaf index and the WANSC could be considered.

On the other hand, by applying the results proposed in Sections 2∼4 and Liao et al. [9], it is shown that the weighted Banzhalf index and the WANSC could be characterized by means of some reasonable properties. Furthermore, the dynamic processes for these two weighted solutions are described that lead the players to these two weighted solutions, starting from an arbitrary efficient payoff vector. Therefore, these two weighted solutions could provide "optimal utilities allocation mechanisms" for warehouse management systems, in the sense that this warehouse organization can get payoff from each combination of all players under TU situation.

In the following, we provide an application with real data as follows. Let $(N, v) \in G$ with $N = \{i, j, k\}$ and $w : N \to R$ with $w(i) = 3$, $w(j) = 5$, $w(k) = 2$. Further, let $v(N) = 9$, $v(\{i\}) = 5$, $v(\{j\}) = -3$, $v(\{k\}) = 4$, $v(\{i, j\}) = 8$, $v(\{i, k\}) = -2$, $v(\{j, k\}) = 3$ and $v(\emptyset) = 0$. By Definitions 1, 2 and 5,

$$
\begin{array}{lll}
\beta_i(N,v) = 6, & \beta_j(N,v) = 11, & \beta_k(N,v) = 1, \\
\overline{\beta_i}(N,v) = 3, & \overline{\beta_j}(N,v) = 8, & \overline{\beta_k}(N,v) = -2, \\
\overline{\beta_i^w}(N,v) = (33/10), & \overline{\beta_j^w}(N,v) = (13/2), & \overline{\beta_k^w}(N,v) = (\text{-}4/5), \\
\mu_i(N,v) = 4, & \mu_j(N,v) = (5/2), & \mu_k(N,v) = 1, \\
\overline{\mu_i}(N,v) = (9/2), & \overline{\mu_j}(N,v) = 3, & \overline{\mu_k}(N,v) = (3/2), \\
\overline{\mu_i^w}(N,v) = (71/20), & \overline{\mu_j^w}(N,v) = (13/4), & \overline{\mu_k^w}(N,v) = (13/10).
\end{array}
$$

5. Conclusions

Weights come up naturally in the framework of utilities allocation. For example, we may be dealing with a problem of utility allocation among investment projects. Then the weights could be associated to the profitability of the different projects. Weights are also included in contracts signed by the owners of a condominium and used to divide the cost of building or maintaining common facilities. Another example is data or patent pooling among firms where the size of the firms, measured for instance by their market shares, are natural weights. Therefore, we adopt weight functions to propose the weighted Banzhaf index. In order to present the rationality of the weighted Banzhaf index, we characterize the weighted Banzhaf index by means of the efficiency-average-reduction. Based on excess functions, alternative formulations of the weighted Banzhaf index and the

WANSC are proposed to provide alternative viewpoints for the weighted Banzhaf index and the WANSC respectively. By applying excess functions, we also define correction functions to propose dynamic processes for these two weighted solutions. One should compare our results with related pre-existing results:

- The weighted Banzhaf index and related results are introduced initially in the framework of standard TU games.
- The dynamic result for the WANSC does not appear in the framework of standard TU games.
- Inspired by Maschler and Owen [12], we propose dynamic processes for the weighted Banzhaf index and the WANSC. The major difference is that our correction functions are based on "excess functions", and Maschler and Owen's [12] correction function is based on "reduced games".

The results proposed in this paper raise two questions:

- Whether there exist weighted modifications and related results for some more solutions.
- Whether there exist different formulations and related results for some more solutions.

These issues are left to the readers.

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Department of Adult and Continuing Education, National Chung Cheng University, Taiwan.

E-mail: aduhcw@ccu.edu.tw

Major area(s): Mathematical economics, adult and continuing education.

Graduate School of Technological and Vocational Education, National Yunlin University of Science and Technology, Taiwan.

E-mail: tku113366@gmail.com

Major area(s): Mathematical economics, technological and vocational education.

Graduate Institute of Educational Administration, National Pingtung University, Taiwan.

E-mail: educator@mail.nptu.edu.tw

Major area(s): Mathematical economics, technological and vocational education.

Department of Applied Mathematics, National Pingtung University, Taiwan.

E-mail: twincos@ms25.hinet.net

Major area(s): Game theory, optimization, operational research, mathematical economics.

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