International Journal of Information and Management Sciences
International Journal of Information and Management Sciences
**28** (2017), 299-316. DOI:10.6186/IJIMS.2017.28.4.1

# Interactive Multiple Criteria Decision Making for Large-Scale Multi-Objective Optimization Problems

*Janusz Miroforidis*

Systems Research Institute, Polish Academy of Sciences

**Abstract**

Despite the rapid development of optimization techniques, there are still practical multi-objective optimization problems hard to solve, e.g., the large-scale portfolio selection or intensity modulated radiation therapy planning. An effective search among potential decisions to such problems can be time consuming or even beyond allotted limits.

To account for this, we propose an interactive multiple criteria decision making scheme with a mix of exact and approximate optimization methods. In that concept, a relatively small set of efficient solutions, so-called shell, is derived by an exact method before the decision making process begins. A shell provides for lower and upper bounds on values of objective functions of efficient decisions and such bounds are easily calculable. During the interactive-iterative decision process such bounds are calculated for decisions corresponding to the decision maker's temporal preferences. Such bounds serve in the decision making process as replacements for the exact values of the objective functions. Bounds stemming from a shell, if not tight enough to conduct the decision process, can be strengthened by lower bounds provided by so-called lower shells, i.e., sets of feasible decisions approximating the set of efficient decisions, derivable by a population based (inexact) method.

We illustrate the operations of the scheme on a selected test problem.

*Keywords:* Multiple criteria decision making, large-scale multi-objective optimization, approximate computations, Pareto frontier navigation.

## 1. Introduction

In almost all practical decision problems, the decision maker (DM) has to evaluate decision variants (decision alternatives or decisions), considering more than one criterion (objective function) at the same time. Many of them are difficult to solve without the assistance of computer methods. For example, the choice of "the best" radiation therapy plan (assuming that the decision problem is formulated with multiple criteria) for a patient is practically impossible without the use of computers and optimization techniques (see Küfer et al. [20]). Such decision problems occur in many areas, e.g., engineering, health care, retailing, financial investments. They are called multi-objective optimization (MOO) problems. In our work, we focus on complex MOO problems, where the use of optimization techniques is necessary to solve them.

In order to solve an MOO problem, among all its Pareto solutions (efficient decision variants or simply — efficient decisions) the DM has to select one Pareto solution that best fits his/her preferences. We make a rational assumption, that only Pareto solutions (the Pareto set) are the DM's focus. The DM does make an evaluation of Pareto solutions (elements of the decision space) by their outcomes (images of Pareto solutions in the outcome space, specified by values of criteria functions — the Pareto frontier). For complex MOO problems, the Pareto set is usually numerous (for combinatorial optimization problems) or infinite (for continuous optimization problems), and the DM needs some support in navigating over the Pareto frontier to point out its interesting regions, and eventually make the decision. The decision is made, when the DM selects exactly one Pareto solution.

By years, many multiple criteria decision making (MCDM) methods have been proposed to solve complex MOO problems (see Branke et al. [3]; Miettinen [24]). Among them, so-called interactive MCDM methods seem to be most rational, as they fully support Herbert Simon's decision making scheme (see Kaliszewski et al. [17]). In these methods, only a number of Pareto solutions is presented to the DM for the evaluation during the decision making process. In this process, his/her preferences are revealed gradually, step by step (by the interaction between the DM and the decision problem, and accumulating the knowledge about the problem). The process terminates when the DM selects his/her the most preferred Pareto solution (the most preferred decision variant). A decision about terminating the process is a sovereign decision of the DM. Therefore, the number of Pareto solutions the DM wish to evaluate, in order to reveal his/her preferences, is *a priori* unknown. One of the most known interactive MCDM method is the reference points method (see Wierzbicki [32]).

In the interactive MCDM methods, the derivation of even a relatively small subset of the Pareto set may be problematic for difficult to solve MOO problems. These difficulties may by caused by, e.g., a large number of decision variables or criteria functions, a shape of the feasible set (determining the set of feasible decisions), time needed to calculate the values of criteria functions for a given decision variant (see Shan, Wang [29]). To solve such problems one may need a specific (maybe costly) optimization software or a help of third-parties offering commercial optimization services. In general, solving such difficult problems might be costly.

An MOO problem for which the cost of the derivation of a subset of its Pareto set cannot be ignored during the decision making process, we call further a *large-scale MOO problem*. We want to emphasize, that in the interactive MCDM methods, Pareto solutions are determined by solving a number of single-objective optimization problems using so-called scalarization techniques (see, e.g., Miettinen [24]). Such (exact) computations may be costly, which should be taken by the DM into account in case of large-scale MOO problems, especially, when he/she wishes to evaluate many Pareto solutions during the interactive decision making process.

An alternative to determining (exact) Pareto solutions in interactive MCDM methods, used to solve large-scale MOO problems, is to use approximate (inexact) methods, e.g., evolutionary single-objective optimization (ESO). ESO methods belong to the broader class — heuristic single-optimization optimization (HSO) methods. Many HSO methods have been proposed to solve single-optimization problems with constraints, to name a few: genetic algorithms (see Goldberg [9]; Michalewicz [23]), the tabu search method (see Glover, Laguna [8]), particle swarm optimization (see Kennedy, Eberhart [19]). To derive an approximate Pareto solution, the single-objective optimization problem should be solved by a selected HSO method. The main advantage of HSO methods is their versatility — they work with any kind of objective and constraint functions. HSO algorithms are relatively easy to implement. Using them to solve practical single-optimization problems is relatively cheap, but they derive (in general) only approximate solutions. This is the main drawback of these methods — one is never sure how far the exact solution is. Even if they work well for sets of the test problems (benchmarks), there is no guarantee that they give a good (in a sense) approximation of the exact solution to a given problem. In other words, the DM may reveal his/her preferences during the decision making process, evaluating only (maybe rough) approximations of Pareto solutions, found by the used HSO method. This may, however, result in making a bad decision.

In the interactive MCDM methods, also heuristic multi-objective optimization (HMO) solutions are used (counterparts of mentioned above HSO methods), e.g., evolutionary multi-objective optimization (EMO) (see Branke et al. [3]). EMO methods are used to derive approximations of potentially interesting for the DM parts of the Pareto frontier. Such preference driven EMO methods allow to derive, for the evaluation by the DM, a number of non-dominated solutions. It allows to avoid the derivation of an approximation of the whole Pareto frontier. Incorporation of the DM's preferences in EMO algorithms has been proposed in many works during the last decade (see, e.g., Filatovas et al. [7]; Jain, Deb [10]; Ruiz et al. [28]). An example on how to control the behaviour of EMO algorithms by the DM's preferences has been proposed in, e.g., Thiele et al. [31]. A practical application of the preference-based EMO algorithm, called R-NSGA-2, to solve an engineering planning problem is presented in, e.g., Mohammadpour et al. [26]. However, due to the use of approximate computations, only elements located maybe not to close to the Pareto frontier are at the disposal of the DM. Therefore, HMO methods (in general) have the same drawback as HSO methods. Despite of this drawback, interactive MCDM methods with HMO methods seem to be good candidates in solving (or helping to solve) large-scale MOO problems. Researchers are aware of usefulness of heuristic methods (especially EMO) in solving large-scale MOO problems, but they also know their limitations (e.g., slow convergence to the Pareto frontier). Therefore, they propose hybrid optimization schemes (or frameworks), as, e.g., in Sindhya [30], where a local search algorithm supports MOO algorithms.

There are also other methods, which are used to limit a number of exact optimization calculations in the literature, e.g., methods with surrogate models (see Jones et al. [11]).

A statistical model (Kriging) is applied in to facilitate feasible solution set sampling in a quest for the Pareto frontier (see Bradley [2]).

In general, all above mentioned heuristic approaches to solve MOO problems have the same drawback. They are not capable to give the DM information on how far from the Pareto frontier efficient outcomes lie. The lack of this information, may bias the decision making process and, in consequence, causes that the DM non effectively searches over the Pareto frontier.

An attempt to limit the number of derived Pareto solutions in the interactive decision making processes, and give the DM information where its outcomes actually lie was taken in Kaliszewski [12]. This method allows to calculate the lower and upper bounds on values of criteria functions of an implicitly given (by so-called vector of concessions) Pareto outcome. In this method, prior to the start of the decision making process, a number of Pareto outcomes (so-called shell) is derived by solving single-optimization problems. On the basis of a shell, the lower and upper bounds on values of criteria functions of an implicitly given Pareto outcome (parametric bounds) can be calculated (with the use of simple, arithmetic formulas). The evaluation of an implicitly given Pareto outcome is made by the DM on the basis of these bounds. The bounds precisely define a region in the outcome space, where the Pareto outcome may be located. To get tighter bounds one can add more Pareto outcomes to the shell, by solving a number of single-optimization problems. In the extreme case, the whole decision making process may be conducted with a relatively small shell, but for the price of using low-precision assessments of implicitly given Pareto outcomes. In case of large-scale MOO problems, the cost of the shell elements' derivation (by costly optimization methods) may be significant, if the DM wishes to use high-precision assessments. The method is universal, because it gives the DM information where Pareto outcomes may lay, regardless of the type of a given MOO problem. There are other methods, which can give the DM such information in the literature, but they are tailored to some classes of MOO problems, e.g., sandwich-type algorithms for convex problems (see Rote [27]), the non-uniform covering method for problems with the Lipschitz type information (see Evtushenko et al. [6]).

In Miroforidis [25] (see also Kaliszewski et al. [16]), a modification of the Kaliszewski's method has been proposed, by replacing shell with a pair of lower shell (consisting of feasible elements of the optimization problem) and upper approximation (consisting of unfeasible elements). Lower shells and upper approximations are derived by EMO methods, hence the necessity of (exact) solving optimization problems during the decision process has been eliminated. This method was used, e.g., to solve mechanical design problems (Kaliszewski et al. [18]). However, there may exist MOO problems for which upper approximations (a base for upper bounds calculations) do not exist (see Kaliszewski, Miroforidis [15]). To overcome this barrier, we propose the modification of this method, which may be used to solve large-scale MOO problems, taking into account the cost of exact optimization calculations. In the proposed method, parametric bounds of implicitly given Pareto outcomes are also used, but the upper bounds are calculated with the use of shells (derived by exact computations), and the lower bounds — with the use of shells or lower shells (derived by approximate EMO computations). The improvement of bounds' quality may be achieved by smart modifications of the lower shell, used in the decision making process. The only assumption in this method is as follows: only a relatively small shell is to be derived (the cardinality of the shell is limited by a given budget for exact optimization calculations) prior to the start of a decision making process.

The hybrid method of solving large-scale MOO problems we propose in this work, is a mix of exact and approximate methods. We use popular EMO computations, but we also give the DM a valuable information about the so-called *optimality gap*, so he/she is able to control the decision making process, on the basis of precisely defined assessments of efficient outcomes, and — indirectly — control the behaviour of used EMO algorithm. Therefore, our work fits well with current trends in hybridization of EMO algorithms and incorporation of the DM's preferences in these class of algorithms. The use of so-called *two-sided Pareto frontier approximations* (derived by exact and inexact methods) during the decision making process we propose, is a novel approach, especially when the exact optimization cost should be taken into account.

The paper is organized as follows. In the next section, we present a generic multiple criteria decision making methodology. In Section 3, the concept of parametric bounds on components of implicitly given Pareto outcomes is presented. In Section 4, we present our interactive MCDM scheme of solving large-scale MOO problems with the use of parametric bounds. In Section 5, we illustrate how the method works on a selected test problem. In Section 6, we discuss the practical aspects of the presented method as well as its limitations. The last section contains a summary of results and possible future works.

## 2. The Multi-Objective Methodology

Let $x$ denote a (decision) *variant* (solution), $X$ a space of variants (*decision space*), $X_0$ a set of *feasible variants*, $X_0 \subseteq X$ . Then the multi-objective optimization problem is:

$$\begin{aligned} & vmax\ f(x) \\ & \text{subject to:} \\ & x \in X_0, \end{aligned} \tag{2.1}$$

where $f : X \to \mathbb{R}^k$, $f = (f_1, \dots, f_k)$, $f_i : X \to \mathbb{R}$, $i = 1, \dots, k$, $k \geq 2$, are *objective functions* (*criteria functions* or simply *criteria*); *vmax* denotes the operator of deriving all *efficient* (as defined below) *variants* in $X_0$. Let $Z = f(X_0) = \{f(x) \mid x \in X_0\}$. Elements of set $Z$ are called *outcomes*, and space $\mathbb{R}^k$ — the *outcome space*.

We say that variant $x$ dominates variant $v$, $x, v \in X$, if $f_i(x) \geq f_i(v)$, $i = 1, \dots, k$, and $f_j(x) > f_j(v)$ for some $j \in \{1, \dots, k\}$. We call variant $x$ *dominating variant* (and its outcome — *dominating outcome*), and variant $v$ — *dominated variant* (and its outcome — *dominated outcome*). Any non-empty subset $V$ of $X$ composed of variants which are not dominated in $V$ we call *non-dominated set of variants* (and its image in the outcome space — *non-dominated set of outcomes*).

Variant $\bar{x}$ of $X_0$, is *efficient* (or *Pareto optimal*), if $f_i(x) \geq f_i(\bar{x})$, $i = 1, \ldots, k$, $x \in X$, implies $f(x) = f(\bar{x})$. Outcomes of efficient variants (non-efficient variants) are called *efficient outcomes* (*non-efficient outcomes*). The set of all efficient variants in set $X_0$ we call *Pareto set*, and its image in the outcome space — *Pareto frontier*.

It is a well-established result (see Ehrgott [5]; Kaliszewski [13]; Miettinen [24]) that variant $\bar{x}$ is efficient (actually, variant $\bar{x}$ is *properly efficient*, for a formal treatment of this issue see Ehrgott [5]; Kaliszewski [13]; Miettinen [24]), if and only if, it solves the optimization problem

$$\min_{x \in X_0} \max_i \lambda_i[(y_i^* - f_i(x)) + \rho e^k(y^* - f(x))], \tag{2.2}$$

where $\lambda_i > 0$, $i = 1, \ldots, k$, $e^k = (1, \ldots, 1)$ and $y^*$ is such that $y_i^* > f_i(x)$, $i = 1, \ldots, k$, $x \in X_0$.

By the "only if" part of this result no efficient variant is excluded from being derived by solving an instance of optimization problem (2.2). In contrast to that, maximization of a weighted sum of objective functions over $X_0$ does not possess, in general (and especially in the case of problems with discrete variables), this property.

Besides the potential ability to derive each efficient variant, optimization problem (2.2) provides for an easy and intuitive capture of the DM's preferences. Observe that element $\hat{y}$, where $\hat{y}_i = \max_{x \in X_0} f_i(x)$, $i = 1, \ldots, k$, represents maximal values of objective functions which can be attained if they are maximized separately.

To assists the DM in the search for the *most preferred variant* one can employ the optimization problem (2.2). Here we assume the minimum of the DM rationality, namely we assume that the decision maker prefers an efficient variant to a non-efficient one.

Suppose that an element $x \in X_0$ such that $\hat{y} = f(x)$ does not exists which is rather a standard with conflicting criteria (otherwise, $x$ is the most preferred variant). Then, the DM knows that whatever efficient variant he/she selects, he/she has to compromise on values of objective functions $f_i$ with respect to values $\hat{y}_i$, $i = 1, \ldots, k$. Due to some formal reasons (see formulas (2.3) and (2.4)), we assume that the decision maker has to compromise on values of objective functions $f_i$ with respect to values $y_i^* = \hat{y}_i + \varepsilon$, $i = 1, \ldots, k$, $\varepsilon > 0$. With arbitrary small values of $\varepsilon$, the difference between $y^*$ and $\hat{y}$ is, especially in practical applications, negligible. He/she can define his/her acceptable compromises on values $y_i^*$, $i = 1, \ldots, k$, and search for an efficient variant which corresponds to this compromise in three ways:

1. providing a *vector of concessions* $\tau$,
2. providing a reference point $y^{ref}$,
3. providing weights $\lambda_i$, $i = 1, \ldots, k$.

Way 1. Components of a vector of concessions $\tau > 0$ ($\tau \in \mathbb{R}^k$) specify concessions the DM accepts to make with respect to $y_i^*$, $i = 1, \ldots, k$. Components of vector $\tau$ can be defined in absolute values ("the DM is willing to make a concession of $z_i$ units on the value $y_i^*$, $i = 1, \ldots, k$") or in relative values ("the DM is willing to make a concession of $z_i$ per cent on the value $y_i^*$, $i = 1, \ldots, k$").

Way 2. A reference point $y^{ref}$ ($y^{ref} \in \mathbb{R}^k$, $y_i^{ref} < y_i^*$, $i = 1, \ldots, k$), (it is irrelevant whether there exists an element $x \in X_0$ such that $y^{ref} = f(x)$ or not) specifies explicitly a compromise between values of objective functions $f_i$ with respect to values $y_i^*$, $i = 1, \ldots, k$, which the DM regards as agreeable. A reference point specifies indirectly a vector of concessions:

$$\tau_i = y_i^* - y_i^{ref}, \quad i = 1, \ldots, k. \tag{2.3}$$

Way 3. An experienced DM can define a vector of concessions $\tau$ in terms of weights $\lambda_i > 0$, $i = 1, \ldots, k$, in optimization problem (2.2). Vector of concessions $\tau$ and vector of weights $\lambda$ are related by formula (2.4).

The optimization problem (2.2) if solved with

$$\lambda_i = (\tau_i)^{-1}, \quad i = 1, \ldots, k, \tag{2.4}$$

has the following property:

- it finds an efficient variant $x$ such that $f(x)$ is on half line $y = y^* - t\tau$, $t \geq 0$, whenever such a variant exists;
- otherwise, it finds an efficient variant $x$ such that

$$\max_i \lambda_i[(y_i^* - f_i(x)) + \rho e^k(y^* - f(x))] = \max_i \lambda_i[(y_i^* - \tilde{y}_i) + \rho e^k(y^* - \tilde{y})],$$

  where $\tilde{y}$ is on half line $y = y^* - t\tau$, $t \geq 0$.

## 3. Parametric Bounds on Components of Efficient Outcomes

The concept of parametric bounds on efficient outcomes' components is crucial to the interactive decision making scheme for MOO problems, proposed in the next section. Below, we give a short description of this concept (for more details see: Kaliszewski [12]; Kaliszewski [13]; Kaliszewski et al. [14]).

Given vector of concessions $\tau$ (the search direction), let us denote by $y(\tau)$ an efficient outcome which would be derived if optimization problem (2.2) was solved with $\lambda_i = (\tau_i)^{-1}$, $i = 1, \ldots, k$. We call $y(\tau)$ *implicit efficient outcome* (implicit Pareto outcome). Element $x(\tau) \in X_0$, $f(x(\tau)) = y(\tau)$, we call *implicit efficient variant* (implicit Pareto variant). We call *shell* a finite non-empty set of efficient outcomes, derived by solving a number of optimization problems (2.2) for different $\lambda$ vectors. We call *lower shell* a finite non-dominated set of feasible outcomes. Elements of the lower shell can be derived with the use of approximate optimization methods.

For given vector of concessions $\tau$, lower shell $S_L$, and shell $S$, parametric bounds on components of implicit efficient outcome $y(\tau)$ are given by inequalities:

$$L_i(S_L, \tau) \leq y_i(\tau) \leq U_i(S, \tau), \quad i = 1, \ldots, k. \tag{3.1}$$

Formulas for calculating bounds $L_i(S_L, \tau)$ and $U_i(S, \tau)$ are relatively simple. They contain only arithmetic operators and operators of deriving minimal and maximal elements in a set of numbers. In order to obtain the correct values of upper bounds, one has to use shells. It is worth observing, that any shell is also a lower shell, hence the values of lower bounds can be calculated with the use of shells. We call the lower and upper bounds on components of $y(\tau)$ *assessment of* $y(\tau)$. The idea of parametric bounds for $k = 2$ is illustrated in Figure 1. The solid line is a Pareto frontier, circles represent a shell's elements, and dots — implicit efficient outcome $y(\tau)$. The search direction (depending on $y^*$ and $\tau$) is represented by a dashed line. The corners $L$ and $U$ of the rectangle represent, respectively, lower and upper bounds on components of $y(\tau)$, and the rectangle itself represents a region, where outcome $y(\tau)$ is located.

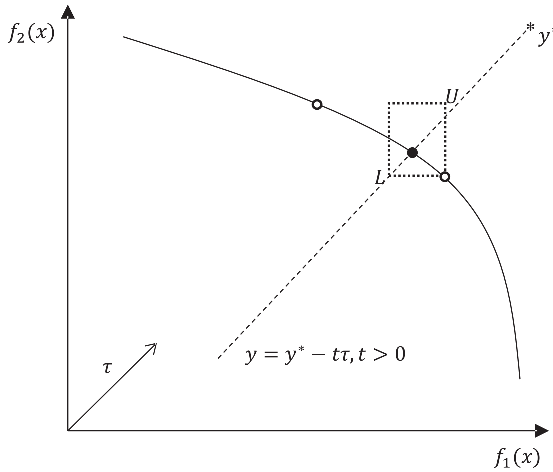


Figure 1: An illustration to the concept of parametric bounds.

For given shell $S$ and vector of concessions $\tau$, to get better upper bounds on values of $y_i(\tau)$ one has to add new elements to $S$. For given lower shell $S_L$ and vector of concessions $\tau$, to get better lower bounds on values of $y_i(\tau)$ one can do it by adding new elements to $S_L$ and eliminating dominated elements in this new set. Direct observation from the lower bounds formulas (see Kaliszewski et al. [14]) is as follows. Given lower shell $\{s\}$ and outcome $w$ for which inequality $\max_i \lambda_i[(y_i^* - w_i) + \rho e^k(y^* - w)] < \max_i \lambda_i[(y_i^* - s_i) + \rho e^k(y^* - s)]$, with $\lambda$ defined by (2.4) holds, then $L_i(\{s\}, \tau) < L_i(\{w\}, \tau) \leq y_i(\tau)$, $i = 1, \ldots, k$.

Assuming that shell $S$ is given and cannot be changed, obtaining tighter bounds (3.1) for vector of concessions $\tau$ is possible only by modifications of given lower shell $S_L$. It follows from the lower bounds formulas, that by the derivation of a set of new non-dominated feasible outcomes $P$, located inside the $k$-dimensional rectangle defined by bounds $L_i(S_L, \tau)$, $U_i(S, \tau)$, and forming new lower shell $\bar{S}_L$ from sets $S_L$ and $P$, one may obtain new (better) lower bounds, such that $L_i(S_L, \tau) < L_i(\bar{S}_L, \tau) \leq y_i(\tau)$, $i = 1, \ldots, k$.

To derive set $P$, any approximate MOO algorithm may be used. The narrowing of the search space can be done by adding $k$ additional constraints $L_i(S_L, \tau) \leq f_i(x)$ to the set of constraints defining set $X_0$. We use this fact in the next section, as a hint for an EMO algorithm to derive elements of the lower shell which may provide with better lower bounds for the given vector of concessions. The idea of such an improvement of lower bounds is illustrated in Figure 2. Elements of the shell (being also a lower shell) are represented by circles, and they are a source of lower bounds $L_1$ and upper bounds $U$ of the efficient outcome. After deriving a new element (triangle) of the lower shell, we get better lower bounds $L_2$.

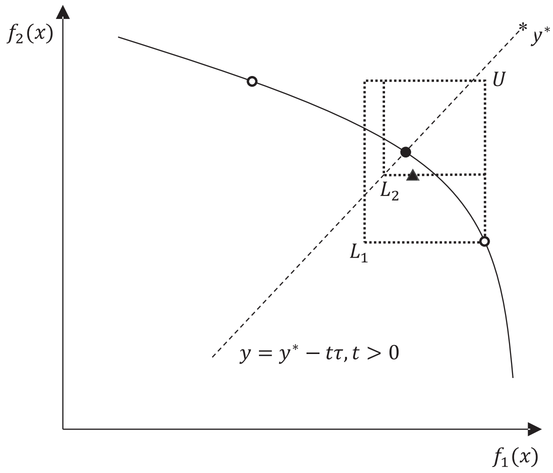


Figure 2: A graphical interpretation of the improvement of lower bounds — the bottom left corner of the inner rectangle ($L_2$) represents better than $L_1$ lower bounds on components of implicit Pareto outcome $y(\tau)$ (represented by a dot).

## 4. The Decision Making Scheme for Large-Scale MOO Problems

In Sections 2 and 3, we presented the general MOO methodology and the concept of parametric bounds on components of implicit Pareto outcomes. Below, we present an interactive MCDM decision making scheme for solving large-scale MOO problems. The scheme is not an algorithm (in a formal sense), because it has no formal stopping rule.

Let us assume that for a given budget for exact optimization calculations, a commercial optimization service (COS) derived a relatively small shell $S$ (e.g., by solving a number of optimization problems (2.2)). The DM would request the shell (not giving the COS any guides) which is a rough representation of the whole Pareto frontier. Let shell $S$ contains (at most $k$) Pareto outcomes needed to calculate element $y^*$. Let $S_L := S$. Let assume also, that an approximate MOO optimizer for cheap modifications of $S_L$ is at disposal of the DM (or supporting him/her an analytical team). The scheme is as follows.

*Step* 1. The DM evaluates the elements of shell $S$. If it contains the most preferred by him/her efficient outcome, the decision process ends. If not, the process continues (Step 2).

*Step* 2. (The body of one iteration — the evaluation of one implicit Pareto outcome.)

1. The DM chooses preference representation by: vectors of concessions ($\tau$), reference points ($y^{ref}$) or weighting vectors ($\lambda$).
2. The DM presents his/her preferences. Regardless of the DM's choice in sub-step 2.1, we denote the implicit Pareto outcome by $y(\tau)$, because appropriate translation mechanisms from reference points or weighting vectors to vectors of concessions do exist — see Section 2.
3. The DM evaluates $y(\tau)$ using its (current) assessment.
4. If the DM is not interested in outcome $y(\tau)$, go to sub-step 2.1.
5. Outcome $y(\tau)$ is the DM's focus. The DM may request a better assessment of $y(\tau)$. Repeat the *LOOP* until: the better assessment of $y(\tau)$ is sufficient for the DM *or* it cannot be derived. *LOOP*: An attempt is made to derive the better assessment of $y(\tau)$ (the approximate MOO optimizer tries to derive new elements of lower shell $S_L$ in the region determined by current lower and upper bounds). If the improvement is not possible, mark that the better assessment of $y(\tau)$ cannot be derived, else, if the assessment of $y(\tau)$ is sufficient for the DM, mark this. *END LOOP*. The improvement of the assessment of $y(\tau)$ may be relatively small (because it can be done only by the improvement of lower bounds), so the DM may refrain from continuing the improvement process assuming that he/she obtained the most accurate assessment, and go to sub-step 2.6.
6. If the assessment of $y(\tau)$ is sufficient and represents the most preferred Pareto outcome, an outcome $z = \arg\min_{y \in S_L} \max_i \lambda_i[(y_i^* - y_i) + \rho e^k(y^* - y)]$ and corresponding (decision) variant is presented to the DM, as the solution to the decision making process, and STOP.
7. If the better assessment of $y(\tau)$ cannot be derived (or the improvement is poor), but, however, the DM decides that it represents the most preferred Pareto outcome, an outcome $z = \arg\min_{y \in S_L} \max_i \lambda_i[(y_i^* - y_i) + \rho e^k(y^* - y)]$ and corresponding (decision) variant is presented to the DM, as the solution to the decision process, and STOP.
8. The DM wishes to evaluate other implicit Pareto outcome, and go to sub-step 2.1.

The presented scheme is an algorithm-like, but a decision about terminating the decision process is a sovereign decision of the DM. He/she can terminate it on its any stage. In sub-steps 6 and 7, by solving optimization problem (2.2), one could derive and outcome $y(\tau)$ (for the vector of concessions, which the best describe the DM's preferences) instead of its representation belonging to the final lower shell.

## 5. Illustrative Example

To illustrate our concept, we selected the problem of investment in $n$ assets (all risky) (see Markowitz [21]; Merton [22]). The well-known bi-objective optimization model for this portfolio investment problem is as follows:

$$\begin{aligned} &\min \tilde{f}_1(x) = x^T Q x && \text{(minimize variance)} \\ &\max f_2(x) = e^T x && \text{(maximize mean)} \\ &\text{subject to } x \in X_0 = \left\{ x \,\middle|\, \begin{matrix} u^T x = 1, \\ x \geq 0 \end{matrix} \right\}, \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$, and $x_i$ is the fraction of the portfolio invested in an $i$-th asset, $Q \in \mathbb{R}^{n \times n}$ is the covariance matrix, $e \in \mathbb{R}^n$ is the vector of means (expected returns), $u \in \mathbb{R}^n$ is the all-ones vector. The "all capital to be invested" condition is represented by constraint $u^T x = 1$, and constraint $x \geq 0$ prohibits short sells of assets. In model (5.1), the first objective function is to be minimized (variance — a measure of risk is minimized). In previous sections, we have assumed, that all objective functions are to be maximized, so we have to maximize objective function $f_1(x) := -\tilde{f}_1(x)$. This substitution is pure technical — we aim to maximize negative value of variance.

We filled model (5.1) with data (vector $e$, matrix $Q$) of the 3-rd portfolio optimization problem with $n = 89$ assets taken from the OR-Library (see Beasley [1]). We used the NSGA-II algorithm (see Deb et al. [4]) as our approximate MOO optimization engine, and we launched it on the off-the-shelf laptop (2GHz CPU, 4GB RAM, Microsoft Windows 10 OS).

We illustrate the concept, by conducting the simulated decision making process (with the fictitious DM), according to the methodology presented in the previous section. Let us recall, that the main idea of the proposed method is the navigation over the Pareto frontier evaluating assessments of implicit Pareto outcomes. If the DM wishes better assessments, we try to improve them by amending the lower shell with the use of an approximate MOO solver. In order to make use of assessments during the decision-making process, a (relatively small) shell is to be derived with the use of an exact optimization method, before the process begins. We aim to illustrate all the steps of the proposed decision making scheme.

Let us assume that the COS has derived shell $S$ with 7 Pareto outcomes (also taken from Beasley [1]) which roughly represent the Pareto frontier. For all computations below, we assume that $\rho = \varepsilon = 10^{-5}$. We derive element $y^* = (-0.000189, 0.008219)$. Element $y^*$ is a combination of the negative value of variance and mean, unattainable by no element from the feasible set. Elements of shell $S$ and element $y^*$ are presented in Figure 3.

We begin our fictitious decision making process. Below, we assume that $y_1(\tau) := y_{-var}(\tau) = f_1(x(\tau))$ and $y_2(\tau) := y_{mean}(\tau) = f_2(x(\tau))$, where $x(\tau)$ is the solution of optimization problem (2.2), if this problem is solved with vector $\lambda$ corresponding to given vector of concessions $\tau$ (see formula (2.4)).

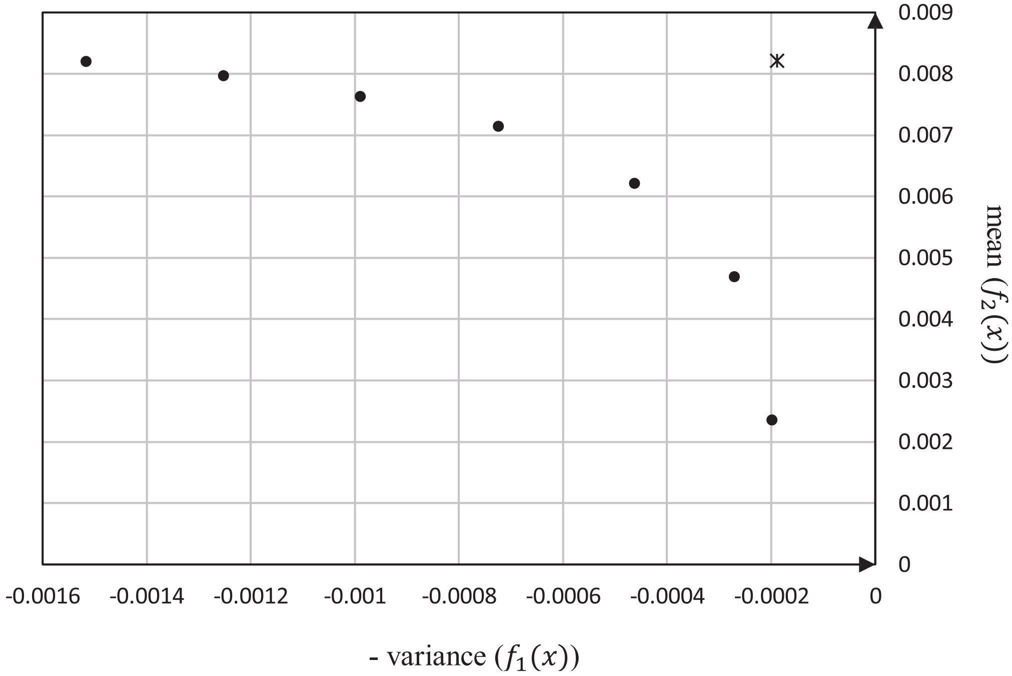


Figure 3: Elements of shell $S$ (dots) and element $y^*$ (a star).

*Step* 1. The DM evaluates the elements of shell $S$ on the basis of their outcomes. It does not contain the most preferred by him/her efficient outcome, so the decision process continues.

*Step* 2. (The navigation over the Pareto frontier with the use of assessments of implicit Pareto outcomes.)

(*Iteration* 1). Suppose that the DM is willing to make concessions on the (impossible) best mean-variance combination $y^*$ and he/she defines such concessions by (the vector of) favourable concessions: (1 risk measure units, 5 monetary units). Hence, $\tau = (1.000, 5.000)$. With this vector of concessions the assessment of $y(\tau)$ is as follows:

$$-0.00059 \leq y_{-var}(\tau) \leq -0.00046$$
$$0.00623 \leq y_{mean}(\tau) \leq 0.00716.$$

Assume that the DM wants a more accurate assessment of $y(\tau)$. To make this, the team supporting the DM runs the NSGA-II algorithm with additional two constraints (based on the above lower bounds), narrowing the search space. The algorithm derives new elements which are added (keeping only non-dominated elements) to lower shell $S_L$. With the new lower shell, the assessment of $y(\tau)$ is as follows:

$$-0.00054 \leq y_{-var}(\tau) \leq -0.00046$$

$$0.00649 \leq y_{mean}(\tau) \leq 0.00716.$$

Suppose that the DM is not satisfied with this portfolio, because of not acceptable variance (risk level). The DM continues the Pareto frontier navigation.

(*Iteration* 2). Suppose that the DM wants to express his preferences by reference points. The DM specifies explicitly a compromise between variance and mean he/she would like to achieve or at least to mimic as closely as possible: $y^{ref} = (-0.0004, 0.006)$. Hence, the search direction is specified by vector of concessions $\tau = (1.058, 11.095)$. With this vector of concessions the assessment of $y(\tau)$ is as follows:

$$-0.00046 \leq y_{-var}(\tau) \leq -0.00027$$
$$0.00535 \leq y_{mean}(\tau) \leq 0.00623.$$

Assume that the DM wants a more accurate assessment of $y(\tau)$. To make this, the team supporting the DM runs the NSGA-II algorithm with additional two constraints (based on the above lower bounds), narrowing the search space. The algorithm derives new elements which are added (keeping only non-dominated elements) to lower shell $S_L$. With the new lower shell, the assessment of $y(\tau)$ is as follows:

$$-0.00043 \leq y_{-var}(\tau) \leq -0.00027$$
$$0.00567 \leq y_{mean}(\tau) \leq 0.00623.$$

Assume that the DM wants again a more accurate assessment of $y(\tau)$. The NSGA-II algorithm derives new elements of the lower shell, and the assessment of $y(\tau)$ is better than previously:

$$-0.00041 \leq y_{-var}(\tau) \leq -0.00027$$
$$0.00585 \leq y_{mean}(\tau) \leq 0.00623.$$

Suppose that (despite of the better assessment) the DM is not satisfied with this portfolio, because of not acceptable variance (risk level). The DM continues the Pareto front navigation.

(*Iteration* 3). The DM has gained a significant knowledge about the decision problem already, and he/she wants to express his/her preferences by reference points again. The DM specifies explicitly a compromise between variance and mean he/she would like to achieve or at least to mimic as closely as possible: $y^{ref} = (-0.00025, 0.004)$. Hence, the search direction is specified by vector of concessions $\tau = (0.308, 21.095)$. With this vector of concessions the assessment of $y(\tau)$ is as follows:

$$-0.00027 \leq y_{-var}(\tau) \leq -0.00020$$
$$0.00257 \leq y_{mean}(\tau) \leq 0.00470.$$

Assume that the DM wants a more accurate assessment of $y(\tau)$. To make this, the team supporting the DM runs the NSGA-II algorithm with additional two constraints (based on the above lower bounds), narrowing the search space. The algorithm derives new elements which are added to lower shell $S_L$. With the new lower shell, the assessment of $y(\tau)$ is as follows:

$$-0.00025 \leq y_{-var}(\tau) \leq -0.00020$$
$$0.00372 \leq y_{mean}(\tau) \leq 0.00470.$$

Suppose that the DM is not satisfied with this portfolio, because of not acceptable mean (expected return). The DM continues the Pareto front navigation.

(*Iteration* 4). The DM wants to express his/her preferences by reference points again. The DM specifies explicitly a compromise between variance and mean he/she would like to achieve or at least to mimic as closely as possible: $y^{ref} = (-0.0003, 0.006)$. Hence, the search direction is specified by vector of concessions $\tau = (0.558, 11.095)$. With this vector of concessions the assessment of $y(\tau)$ is as follows:

$$-0.00037 \leq y_{-var}(\tau) \leq -0.00027$$
$$0.00470 \leq y_{mean}(\tau) \leq 0.00623.$$

Assume that the DM wants a more accurate assessment of $y(\tau)$. To make this, the team supporting the DM runs the NSGA-II algorithm with additional two constraints (based on the above lower bounds), narrowing the search space. The algorithm derives new elements which are added to lower shell $S_L$. With the new lower shell, the assessment of $y(\tau)$ is as follows:

$$-0.00034 \leq y_{-var}(\tau) \leq -0.00027$$
$$0.00525 \leq y_{mean}(\tau) \leq 0.00623.$$

Suppose that the new assessment is accurate for the DM and he/she considers the implicit Pareto outcome, represented by this assessment, as the most preferred one (the most preferred combination of risk and expected return). The solution to the decision making process is outcome $(-0.00034, 0.00525) = \arg\min_{y \in S_L} \max_{i \in \{1,2\}} \lambda_i[(y_i^* - y_i) + \rho e^k (y^* - y)]$ and the corresponding portfolio. Vector $\lambda$ depends on the revealed by the DM vector of concessions $\tau$ according to formula (2.4). The decision making process terminates.

The final stage of the decision making process is presented in Figure 4 (narrowed to the DM's region of interest). The dashed line represents the search direction specified by the last vector of concessions (Iteration 4); shell $S$ is represented by circles, but (all) elements derived by the NSGA-II algorithm — by dots. Circles and dots represent the final lower shell. The larger rectangle represents the assessment of $y(\tau)$ if computed only with the use of the shell, but the smaller one — if computed with the use of the lower shell (only for lower bounds calculations) and the shell (the assessment of $y(\tau)$ calculated in Iteration 4, the last improvement).

Elements of the lower shell, derived by the NSGA-II algorithm, are probably very close to the Pareto frontier (see Figure 4), but we cannot use them as a base for calculations of upper bounds. To obtain the correct values of upper bounds one has to use only

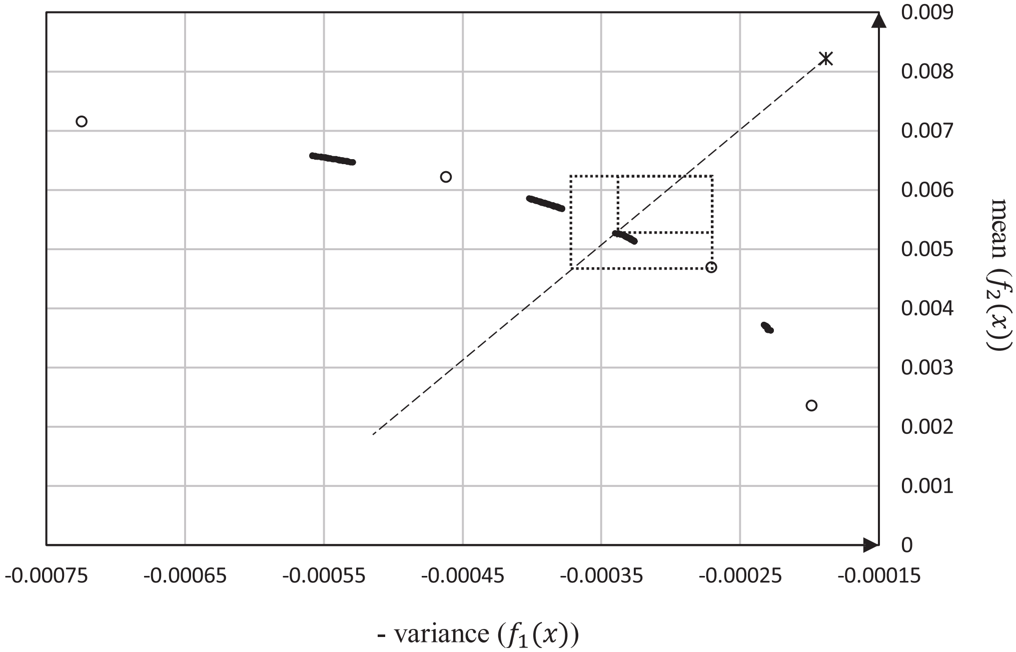


Figure 4: The final stage of the decision making process.

elements of the shell. To get better upper bounds the derivation of new elements of the Pareto frontier is necessary.

The NSGA-II algorithm was launched with the number of generations equal to 400, and the population size equal to 50 (only for the second run of Iteration 2, we used the population size equal to 100, because 50 was not enough to derive any new element of the lower shell). The computation time of each run was below one minute.

## 6. Discussion

As we have seen, the presented illustrative bi-objective optimization problem was not problematic for the NSGA-II algorithm we used. It derived elements of the lower shell (in regions specified by lower bounds) quickly, with relatively small values of its two basic parameters: number of generations, population size. To solve real-life instances of large-scale multi-objective optimization problems, the use of large values of these parameters may be necessary. In general, for any heuristic multi-objective optimization method, wchich may be used in the proposed decision scheme, the derivation of new elements of the lower shell may be time consuming. In the extreme case, if new constraints (let us recall, specified by lower bounds) are to strong for the selected method, no new element would be derived. It depends on the optimization problem or the current stage of the decision making process (the DM requests better and better assessments). A team supporting the DM and delivering him/her new assessments of implicit Pareto outcomes can switch to another approximate method during the decision making process, if the selected one is not capable to derive a new lower shell providing better lower bounds.

In the proposed decision making scheme, the shell does not change during the decision making process. If the DM is not able to make a decision on the basis of poor assessments, he/she would increase the budget for exact computations. After delivering a new shell by the commercial optimization company, the decision process would begin again. How good assessments are necessary to evaluate implicit Pareto outcomes? It depends on the DM and his/her accumulated knowledge about the decision problem he/she solves. Let us recall that at the end of the decision making process, an outcome (belonging to the lower shell) and corresponding (decision) variant are presented to the DM. However, at the end of the process final vector of concessions (corresponding to the DM's preferences) is also revealed. It is possible to compute the efficient variant, specified by this vector, with the use of exact optimization calculations. Such element could be finally presented to the DM.

The main advantage of the presented decision making scheme is the possibility of making use of approximate optimization methods, but with precisely defined lower and upper bounds on implicit Pareto outcomes. Limitations of our method are as follows: 1) for real-life MOO problems, approximate computations may be time consuming, and unacceptable for the DM; 2) we assumed that the DM is supported by its analytical team — it may also be costly, especially if the team becomes deeply involved in the decision making processes (e.g., software development and parametrization).

## 7. Concluding Remarks

The generic interactive multiple criteria decision making scheme for large-scale multi-objective optimization problems has been presented. In the scheme, a mix of exact and approximate optimization methods is used. In the proposed method, a budget for exact optimization calculations is established, therefore only a limited number of elements of the Pareto frontier are to be derived during the decision making process. Such efficient outcomes are derived prior to the start of the decision process. During the Pareto frontier navigation, the DM reveals his/her temporal preferences evaluating only assessments of implicit Pareto outcomes, given by vectors of concessions. Such assessments are easy to calculate with the use of relatively simple formulas. The use by the DM of mathematicaly defined assessments of implicit Pareto outcomes (instead of their exact values, which derivation might be costly), is the main advantage of the presented decision scheme in the context of large-scale multi-objective optimization. Not expensive, heuristic multi-objective optimization methods, are used in the proposed scheme to improve the quality of assessments. A heuristic multi-objective optimization solver is controlled by the DM's preferences, revealed with the help of assessments.

A simple, but illustrative example of the portfolio selection problem, has been presented. The ficititious decision making proces has been conducted to illustrate all the steps of the proposed decision scheme. As a heuristic multi-objective optimization method, the NSGA-II algorithm was used. The practical aspects of the proposed method, have also been discussed.

Further work will cover the following two topics. First, we will concentrate on the issues of solving real-life large-scale multi-objective optimization problems, and testing approximate methods as a tool for the improvement of assessments. Second, we would like to check, how a real DMs make an evaluation of implicit Pareto outcomes with the use of lower and upper bounds on values of theirs components.

## References

[1] Beasley, J. E. (1991). *OR-Library*, `http://people.brunel.ac.uk/~mastjjb/jeb/info.html`.

[2] Bradley, P. J. (2014). *Kriging-Pareto Front Approach for the Multi-Objective Exploration of Metamaterial Topologies*, Progress In Electromagnetics Research, Vol.39, 141-150.

[3] Branke, J., Deb, K., Miettinen, K., Słowiński R., (eds.) (2008). Multiobjective Optimization — Interactive and Evolutionary Approaches, Springer-Verlag, Berlin.

[4] Deb, K., Agrawal, S., Pratap, A. and Meyarivan, T. (2002). *A fast and elitist multi-objective genetic algorithm: NSGA-II*, IEEE Transactions on Evolutionary Computation, Vol.6, No.2, 182-197.

[5] Ehrgott, M. (2005). Multicriteria Optimization, Springer-Verlag, Berlin.

[6] Evtushenko, Yu. G. and Posypkin, M. A. (2013). *Nonuniform Covering Method as Applied to Multicriteria Optimization Problems with Guaranteed Accuracy*, Computational Mathematics and Mathematical Physics, Vol.53, No.2, 144-157.

[7] Filatovas, E., Lančinskas, A., Kurasova, O. et al. (2016). *A preference-based multi-objective evolutionary algorithm R-NSGA-II with stochastic local search*, Central European Journal of Operations Research, `https://doi.org/10.1007/s10100-016-0443-x`.

[8] Glover, F. and Laguna, M. (1997). Tabu Search, Kluwer Academic Publishers, Boston.

[9] Goldberg, D. E. (1989). Genetic Algorithms in Search, Optimization and Machine Learning, Addison-Wesley Publishing Co.

[10] Jain, H. and Deb, K. (2014). *An evolutionary many-objective optimization algorithm using reference-point based non-dominated sorting approach, part II: handling constraints and extending to an adaptive approach*, IEEE Trans Evolutionary Computations, Vol.18, No.4, 602-622.

[11] Jones, D. R., Schonlau, M. and Welch, W. J. (1998). *Efficient global optimization of expensive black-box functions*, Journal of Global Optimization, Vol.13, 455-492.

[12] Kaliszewski, I. (2004). *Out of the mist - towards decision-maker-friendly Multiple Criteria Decision Making support*, European Journal of Operational Research, Vol.158, No.2, 93-307.

[13] Kaliszewski, I. (2006). Soft Computing for Complex Multiple Criteria Decision Making, Springer, New York.

[14] Kaliszewski, I., Miroforidis, J. and Podkopaev, D. (2009). *A low-complexity approach to supply chain management*, In: Proceedings of the 2009 International Conference in Management Sciences and Decision Making.

[15] Kaliszewski, I. and Miroforidis, J. (2012). *Real and virtual Pareto set upper approximations*, In: Multiple Criteria Decision Making '11, Trzaskalik T., Wachowicz T. (eds.), The Publisher of the University of Economics in Katowice, Katowice, 121-131.

[16] Kaliszewski, I., Miroforidis, J. and Podkopaev, D. (2012). *Interactive Multiple Criteria Decision Making based on preference driven Evolutionary Multiobjective Optimization with controllable accuracy*, European Journal of Operational Research, Vol.216, No.1, 293-307.

[17] Kaliszewski, I., Miroforidis, J. and Podkopayev, D. (2016). Multiple Criteria Decision Making by Multiobjective Optimization — A Toolbox, Springer, New York.

[18] Kaliszewski, I., Kiczkowiak, T. and Miroforidis, J. (2016). *Mechanical design, Multiple Criteria Decision Making and Pareto optimality gap*, Engineering Computations, Vol.33, No.3, 876-89.

[19] Kennedy, J. and Eberhart, R. (2001). Swarm Intelligence. Morgan Kaufmann Publishers.

[20] Küfer, K.-H., Monz, M., Scherrer, A., Süss, P., Alonso, F., Sultan, A. S. A., Bortfeld, Th., Craft, D. and Thieke, Chr. (2005). Multicriteria optimization in intensity modulated radiotherapy planning, Berichte des Fraunhofer ITWM, 77.

[21] Markowitz, H. M. (1959). Portfolio Selection: Effcient Diversiffication of Investments, Wiley, 1959, Yale University Press, 1970, Basil Blackwell, 1991.

[22] Merton, R. C. (1972). *An analytical derivation of the efficient portfolio frontier*, Journal of Financial and Quantitative Analysis, Vol.7, No.4, 1851-1872.

[23] Michalewicz, Z., (1998). Genetic Algorithms + Data Structures = Evolution Programs (third edition), Springer-Verlag, Berlin.

[24] Miettinen, K. M. (1999). Nonlinear Multiobjective Optimization, Kluwer Academic Publishers, Boston.

[25] Miroforidis, J. (2010). Decision Making Aid for Operational Management of Department Stores with Multiple Criteria Optimization and Soft Computing, PhD Thesis (in Polish), Systems Research Institute, Polish Academy of Sciences, Warsaw.

[26] Mohammadpour, A., Dehghani, A. and Byagowi, Z, (2013). *Using R-NSGA-II in the transmission expansion planning problem for a deregulated power system with wind farms*, International Journal of Engineering Practical Research, Vol.2, No.4, 201-204.

[27] Rote, G. (1992). *The convergence rate of the sandwich algorithm for approximating convex functions*, Computing, Vol.48, 337-361.

[28] Ruiz, A. B., Saborido, R. and Luque, M. (2015). *A preference-based evolutionary algorithm for multiobjective optimization: the weighting achievement scalarizing function genetic algorithm*, Journal of Global Optimization, Vol.62, No.1, 101-129.

[29] Shan, S. and Wang, G. G. (2010). *Survey of modelling and optimization strategies to solve high-dimensional design problems with computationally-expensive black-box functions*, Structural and Multidisciplinary Optimization, Vol.41, No.2, 219-241.

[30] Sindhya, K., Miettinen, K. and Deb, K. (2013). *A hybrid framework for evolutionary multi-objective optimization*, IEEE Transactions on Evolutionary Computation, Vol.17, No.4, 495-511.

[31] Thiele, L., Miettinen, K., Korhonen, P. J. and Molina, J. (2009). *A preference-based evolutionary algorithm for multi-objective optimization*, Evolutionary Computations, Vol.17, No.3, 411-436.

[32] Wierzbicki, A. P. (2000). Reference point methodology, In: Wierzbicki, A.P., Makowski, M., Wessels, J. (eds.), Model-Based Decision Support Methodology with Environmental Applications, 71-89, Kluwer Academic Publishers, Dordrecht.

Department of Intelligent Systems, Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland.

E-mail: janusz.miroforidis@ibspan.waw.pl

Major area(s): Multiple criteria decision making, large-scale multi-objective optimization, soft-somputing.

(Received May 2017; accepted November 2017)